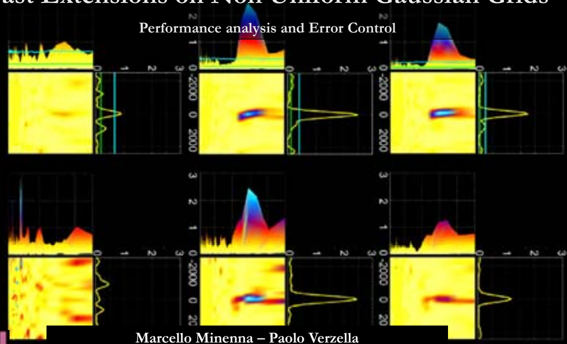
DFT Methods for Option Pricing
Fast Extensions on Non Uniform Gaussian Grids

Performance analysis and Error Control



Marcello Minenna – Paolo Verzella

IX WORKSHOP ON QUANTITATIVE FINANCE
January 24-25, 2008
University of Rome "Tor Vergata"

CONSOB

Syllabus of the presentation

- Review of Option Pricing via DFT
  - FT Pricing formula
  - DFT Convergence to FT
  - Convergence Theorems for Uniform Grids
  - Convergence Theorems for Non Uniform Gaussian Grids
- Fast Option Pricing
  - FFT
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CONSOB

FT Pricing Formulas

European Call Price C_t

Spot Price S_t

$$f_2(\ln S_T, \xi \mid \ln S_0) = \int_{-\infty}^{+\infty} e^{i\xi \ln S_T} q_2(\ln S_T \mid \ln S_0)\, d\ln S_T$$

under risk-neutral measure

A linear direct mapping from Fourier Space

FT

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FT

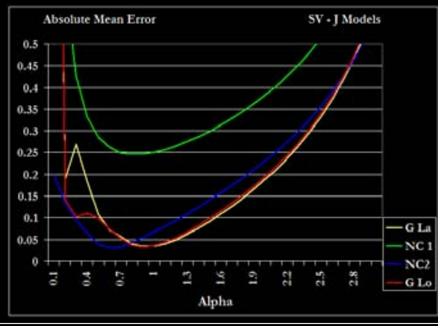
$$C_0 = \frac{e^{-\alpha \ln K}}{\pi} \int_0^{+\infty} \Re\left(e^{i\xi \ln K} \frac{e^{-rT} f_2(\xi - (\alpha+1)i)}{\alpha^2 + \alpha - \xi^2 + i(2\alpha+1)\xi} \right) d\xi$$

CONSOB

FT Pricing Formulas

Accuracy

Absolute Mean Error computed w.r.t. α on an (σ, τ) space

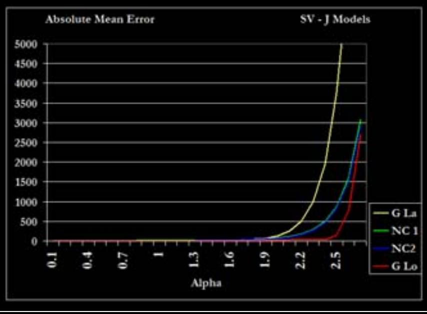


CONSOB

FT Pricing Formulas

Stability

Absolute Mean Error computed w.r.t. α on an Extended (σ, τ) space



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CONSOB

DFT Convergence to FT

Given the General DFT

$$\omega(m) = \sum_{j=0}^{N-1} e^{-i\frac{2\pi}{X}x_j(m-1)} f(x_j) \quad \text{where } m = 1,2,\ldots M$$

CONSOB

Given the General DFT

$$\omega(m) = \sum_{j=0}^{N-1} e^{-i\frac{2\pi}{X}x_j(m-1)} f(x_j) \quad \text{where } m = 1,2,\ldots M$$

$$M \neq N$$

The Convergence Theorem for General DFT's (C Th)

$$\mathcal{F}[f(x)](t_m) = \lim_{N\to\infty} \sum_{j=1}^{N} e^{-i\frac{2\pi}{X}x_j(m-1)} f(x_j, X)$$

$$t_m = \frac{2\pi}{X}(m-1)$$

C_0 via FT

Convergence theorems

$\overrightarrow{C_0}$ via DFT

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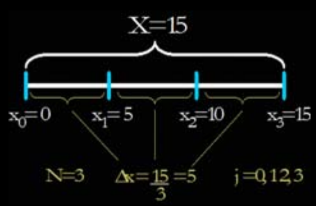
Condition 1

Uniform Discretization Grid



Condition 2

N=M

DFT specialized

$$\omega(n) = \sum_{j=0}^{N-1} e^{-i\frac{2\pi}{X}x_j(n-1)} f(x_j) \quad \text{where } n=1,2,\ldots,N$$

Condition 1 Condition 2

DFT Simplified Formula

$$\omega(n) = \sum_{j=0}^{N-1} e^{-i\frac{2\pi}{N}j(n-1)} f(x_j) \quad \text{where } n = 1,2,\ldots N$$

Nyquist – Shannon Limit (N-S)

$$\mathcal{F}[f(x)](t_n) = \lim_{N\to\infty} \frac{X}{N}\omega(n)$$

$$\{t_n\}_{n=1..\frac{N}{2}} \quad \text{for } N \text{ even}$$

$$\{t_n\}_{n=1..\frac{N+1}{2}} \quad \text{for } N \text{ odd}$$

$$C_0 = \frac{e^{-\alpha \ln K}}{\pi} \int_0^{+\infty} \Re\left(e^{i\xi \ln K} \frac{e^{-rT} f_2(\xi-(\alpha+1)i}{\alpha^2+\alpha-\xi^2+i(2\alpha+1)\xi} \right) d\xi$$

Uniform Discretization Grids for *f*

1. $f(v_{j-1}) = e^{-i(j-1)\eta[\ln S_t - b]}\Psi_0[(j-1)\eta]$
2. $f(v_{j-1}) = e^{-i(j-1)\eta[\ln S_t - b]}\Psi_0[(j-1)\eta]\cdot[3+(-1)^{j+1}-\delta_j-\delta_{N-j}]$

1.
$$C_0=\frac{e^{-\alpha \ln K}}{\pi}\int_0^{+\infty}\Re\left(e^{i\xi \ln K}\frac{e^{-rT}f_2(\xi-(\alpha+1)i}{\alpha^2+\alpha-\xi^2+i(2\alpha+1)\xi}\right)d\xi$$

$$f(v_{j-1})=e^{-i(j-1)\eta[\ln S_t-b]}\Psi_0[(j-1)\eta]$$

+

N-S

$$C_0[\ln K]_u^- \approx \frac{e^{-\alpha[\ln S_t-b+\lambda(u-1)]}}{b}\cdot\Re(\omega(u))$$

2.
$$C_0=\frac{e^{-\alpha \ln K}}{\pi}\int_0^{+\infty}\Re\left(e^{i\xi \ln K}\frac{e^{-rT}f_2(\xi-(\alpha+1)i}{\alpha^2+\alpha-\xi^2+i(2\alpha+1)\xi}\right)d\xi$$

$$f(v_{j-1})=e^{-i(j-1)\eta[\ln S_t-b]}\Psi_0[(j-1)\eta]\cdot[3+(-1)^{j+1}-\delta_j-\delta_{N-j}]$$

+

N-S

$$C_0[\ln K]_u^- \approx \frac{e^{-\alpha[\ln S_t-b+\lambda(u-1)]}}{3b}\cdot\Re(\omega(u))$$

Theorems of Equivalence

The Call Price computed via Convergence Theorem is equal to the Call Price computed via Trapezoid/Simpson Quadrature Rule

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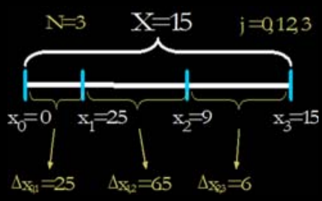
Condition 1

Non Uniform Discretization Grid



Condition 1

Gaussian Grids

Optimal choice of discretization points

Gauss Laguerre

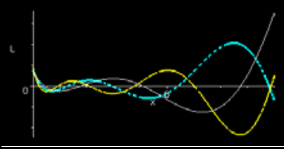
Gander Gautschi

Condition 1

Gaussian Grids

Optimal choice of discretization points

Gauss Laguerre



Zeros of Laguerre Poynomials

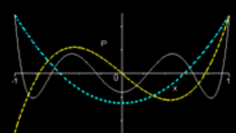
Condition 1

Gaussian Grids

Optimal choice of discretization points

Gauss Lobatto

Zeros of Legendre Poynomials



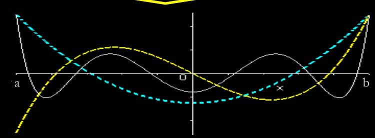
Condition 1

Gaussian Grids

Optimal choice of discretization points

Gander Gautschi

Zeros of rescaled Legendre Poynomials



Condition 2

N≠M

General DFT

$$\omega(m)=\sum_{j=0}^{N-1} e^{-i\frac{2\pi}{X}x_j(m-1)} f(x_j) \text{ where } m=1,2,\dots,2M$$

The Convergence Theorem for General DFT's (C Th)

$$\mathcal{F}[f(x)](t_m) = \lim_{N\to\infty}\sum_{j=1}^{N} e^{-i\frac{2\pi}{X}x_j(m-1)} f(x_j,X)$$

$$t_m = \frac{2\pi}{X}(m-1)$$

$$C_0=\frac{e^{-\alpha\ln K}}{\pi}\int_0^{+\infty}\Re\left(e^{i\xi\ln K}\frac{e^{-rT}f_2(\xi-(\alpha+1)i}{\alpha^2+\alpha-\xi^2+i(2\alpha+1)\xi}\right)d\xi$$

Gaussian Grids for *f*

1. $f(\upsilon_{j-1}) = e^{\left[1+i\left(\frac{M\pi}{a*}-\ln S_t\right)\right]\upsilon_{j-1}}\psi_0(\upsilon_{j-1}) \quad \frac{1}{L_{N+1}(\upsilon_{j-1})L'_N(\upsilon_{j-1})}$

2. $f\left(\frac{1}{2}a(1+\upsilon_{j-1})\right) = \left[e^{-i\left(\frac{1}{2}a(1+\upsilon_{j-1})\right)\left[\ln S_t-\frac{M\pi}{a}\right]}\psi_0\left(\frac{1}{2}a(1+\upsilon_{j-1})\right)\right]\frac{1}{[P_{N-1}(\upsilon_{j-1})]^2}$

1.

$$C_0=\frac{e^{-\alpha\ln K}}{\pi}\int_0^{+\infty}\Re\left(e^{i\xi\ln K}\frac{e^{-rT}f_2(\xi-(\alpha+1)i}{\alpha^2+\alpha-\xi^2+i(2\alpha+1)\xi}\right)d\xi$$

$$f(\upsilon_{j-1}) = e^{\left[1+i\left(\frac{M\pi}{a*}-\ln S_t\right)\right]\upsilon_{j-1}}\psi_0(\upsilon_{j-1}) \quad \frac{1}{L_{N+1}(\upsilon_{j-1})L'_N(\upsilon_{j-1})}$$

\+

C-Th

$$C_0([\ln K]^*_u) \approx -\Re\left[\frac{e^{-a\left(\ln S_t-\frac{M\pi}{a*}+\frac{2\pi}{a*}(u-1)\right)}}{\pi}\frac{1}{N+1}\cdot\omega^*(u)\right]$$

2.

$$C_0=\frac{e^{-\alpha\ln K}}{\pi}\int_0^{+\infty}\Re\left(e^{i\xi\ln K}\frac{e^{-rT}f_2(\xi-(\alpha+1)i}{\alpha^2+\alpha-\xi^2+i(2\alpha+1)\xi}\right)d\xi$$

$$f\left(\frac{1}{2}a(1+\upsilon_{j-1})\right) = \left[e^{-i\left(\frac{1}{2}a(1+\upsilon_{j-1})\right)\left[\ln S_t-\frac{M\pi}{a}\right]}\psi_0\left(\frac{1}{2}a(1+\upsilon_{j-1})\right)\right]\frac{1}{[P_{N-1}(\upsilon_{j-1})]^2}$$

\+

C-Th

$$C_0([\ln K]^*_u) \approx \Re\left[\frac{e^{-a\left(\ln S_t-\frac{M\pi}{a*}+\frac{2\pi}{a*}(u-1)\right)}}{\pi}\frac{1}{N(N-1)}\cdot\omega^*\left(\tfrac{1}{2}a(1+\upsilon_{j-1})\right)\right]$$

Theorems of Equivalence

The Call Price computed via Convergence Theorem is equal to the Call Price computed via Gauss Laguerre/Gander Gautschi Quadrature Rule

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$\overrightarrow{C_t}$ via DFT

allows

Fast Fourier Trasform Algorithms

$\overrightarrow{C_t}$ via DFT

Newton-Cotes

Uniform FFT

$\overrightarrow{C}_t$ via DFT

Gauss

NonUniform FFT

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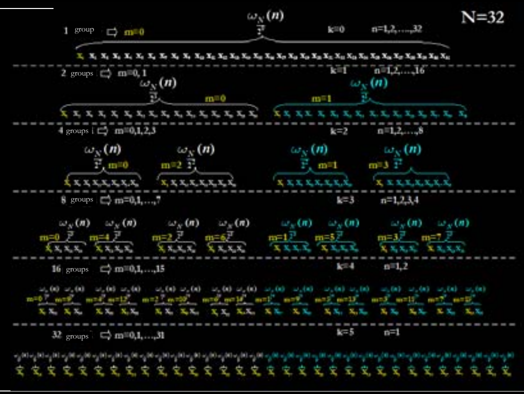
Cooley-Tukey DFT Characterization

$$\omega_2^m(n) = f(x_m) + W_2^{(n-1)} f(x_{m+N/2}) \quad for\ n = 1,2$$

Iterated Bottom – Up for N stages

It gives the FFT Cooley – Tukey Algorithm



FFT Cooley – Tukey Algorithm

The DFT computational cost drops

from

$O(N^2)$

to

$O(N \log_2 N)$

Since the Nyquist – Shannon Limit,
the pricing formulas

via FFT

Give accurate prices
ONLY
Around the Nyquist Frequency

Since the Nyquist – Shannon Limit,
the pricing formulas

via FFT

Give accurate prices
ONLY
Around the Nyquist Frequency

Approx. 25% of prices can be accepted

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Gaussian Gridding

Gaussian Gridding

Step 1

Gaussian Gridding

Step 1

Gaussian Convolution of the non uniformly sampled characteristic function

Gaussian Gridding

Step 1

Gaussian Convolution of the non uniformly sampled characteristic function

$$f_\tau(x) = \sum_{j=0}^{N-1} f(x_j) \sum_{k=-\infty}^{\infty} e^{-\frac{(x_j - x - 2k\pi)^2}{4\tau}}$$

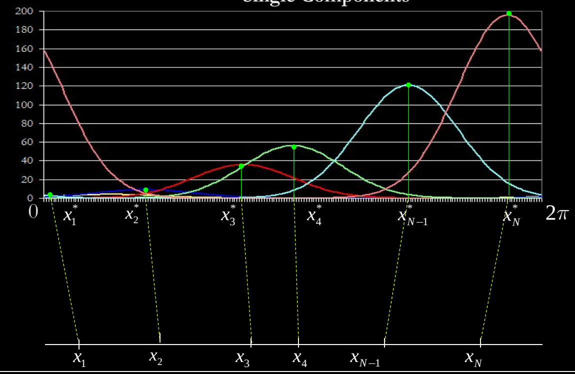
Gaussian Gridding

Step 1

Gaussian Convolution of the non uniformly sampled characteristic function

$$f_\tau(x) = \sum_{j=0}^{N-1} f(x_j) \sum_{k=-\infty}^{\infty} e^{-\frac{(x_j - x - 2k\pi)^2}{4\tau}}$$

Single Components



Gaussian Gridding

Step 1

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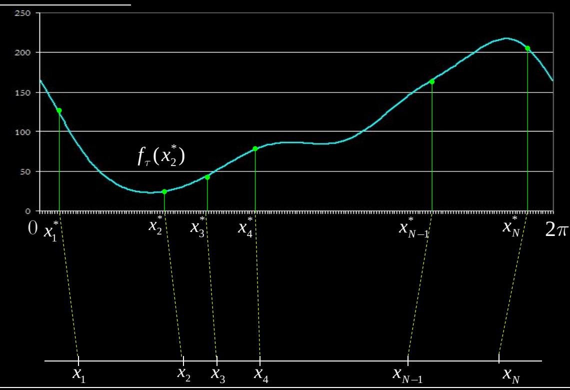


Gaussian Gridding

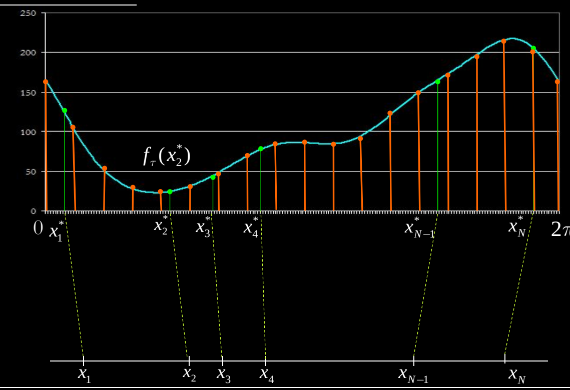
Step 2

Discretization on an uniform oversampled grid of $f_\tau(x)$

$$\widetilde{f}_\tau(y_m) = \sum_{j=0}^{N-1} f(x_j)\, \widetilde{K}(y_m, x_j, \sqrt{2\tau}; 2\pi)$$

where

$$\widetilde{K}(y_m, x_j, \sqrt{2\tau}; 2\pi) = \begin{cases} \sum_{k=-\infty}^{\infty} e^{-\frac{[illegible]}{4\tau}} \\ \text{oppure} \\ \sum_{k=-\infty}^{\infty} e^{-\frac{[illegible]}{4\tau}} \end{cases}$$



Gaussian Gridding

Step 3

Computation of the Fourier Coefficient of $f_\tau(x)$ discretised

$$F_\tau(n) = \lim_{M_\tau \to \infty} \frac{1}{M_\tau} \sum_{m=0}^{M_\tau - 1} \tilde{f}_\tau \left(m \frac{2\pi}{M_\tau} \right) e^{-im\frac{2\pi}{M_\tau}(n-1)}$$

Gaussian Gridding

Step 4

NU-DFT representation of the Fourier Coefficient $F_\tau(n)$

$$\tilde{\omega}(n) = \sqrt{\frac{\pi}{\tau}} e^{n^2 \tau} F_\tau(n)$$

Gaussian Gridding

Step 5

DFT representation of the Fourier Coefficient $F_\tau(n)$

$$F_\tau(n) = \lim_{M_\tau \to \infty} \frac{1}{M_\tau} \omega(n)$$

for $n = 1, 2, \ldots, \frac{M_\tau}{2}$

Gaussian Gridding

Step 6

NU-DFT derivation as a function of DFT

Gaussian Gridding

Step 6

NU-DFT derivation as a function of DFT

$$F_\tau(n) = \lim_{M_\tau \to \infty} \frac{1}{M_\tau} \omega(n)$$

$$\tilde{\omega}(n) = \sqrt{\frac{\pi}{\tau}} e^{n^2 \tau} F_\tau(n)$$

$$\tilde{\omega}(n) = \lim_{M_\tau \to \infty} \sqrt{\frac{\pi}{\tau^*}} e^{n^2 \tau^*} \frac{1}{M_\tau} \omega(n)$$

for $n = 1, 2, \ldots, \frac{M_\tau}{2}$

Gaussian Gridding

Step 7

NU-FFT computation

Gaussian Gridding

Step 7

NU-FFT computation

$$\tilde{\omega}(n) = \lim_{M_\tau \to \infty} \sqrt{\frac{\pi}{\tau^*}} e^{n^2 \tau^*} \frac{1}{M_\tau} \omega(n)$$

FFT

Gaussian Gridding

Step 7

NU-FFT computation

$$\tilde{\omega}(n) = \lim_{M_\tau \to \infty} \sqrt{\frac{\pi}{\tau^*}} e^{n^2 \tau^*} \frac{1}{M_\tau} \omega(n)$$

NU-FFT

FFT

Computational Cost

The major computational cost of the Procedure is the FFT on the oversampled grid

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Choosing the oversampling ratio

$M_\tau = 2M$

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The major computational cost of the Procedure is the FFT on the oversampled grid

Choosing the oversampling ratio

$M_\tau = 2M$

The total cost of the procedure is $\simeq 2M \log 2M$

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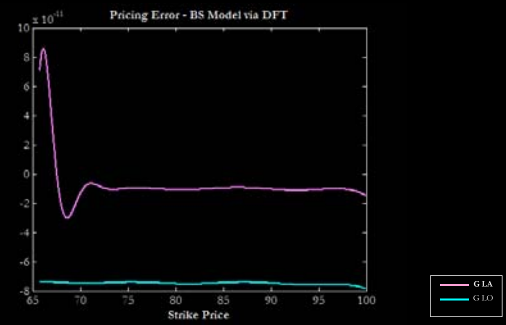
ACCURACY

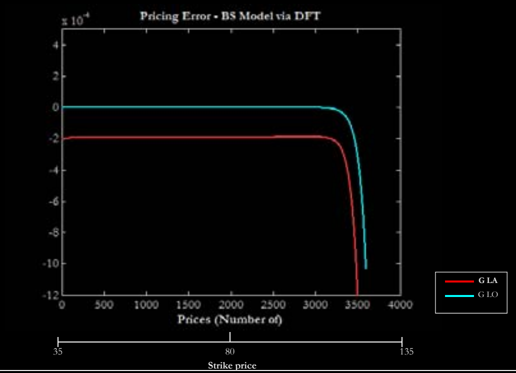
GLA
GLO



2000 Prices computed

STABILITY

GLA
GLO



STABILITY

The error of 90% of prices computed lies in the

STABILITY

The error of 90% of prices computed lies in the

10^{-3}

RANGE OF PRECISION

SPEED

CONSOB

SPEED

the NU – FFT is around
2 time slower than FFT

CONSOB

SPEED

At very low time scales, the
differences disappear

CONSOB

SPEED

At very low time scales, the
differences disappear

	NC2	G – LA	G - LO
FFT	0.01 sec.	N/A	N/A
	NC2	G - LA	G - LO
NU – FFT	0.02 sec.	0.0261 sec.	0.0301 sec.

Computation of 4000 prices on a Centrino 1600Mhz – 2gb RAM
Mean Value over 1000 runs

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- NU – FFT allows the use of Gaussian Grids

CONSOB

- NU – FFT allows the use of Gaussian Grids
- NU – FFT is indifferent to Nyquist _Shannon Limit

CONSOB

- NU – FFT allows the use of Gaussian Grids
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- NU – FFT is at least as accurate as FFT

CONSOB

- NU – FFT allows the use of Gaussian Grids
- NU – FFT is indifferent to Nyquist _Shannon Limit
- NU – FFT is at least as accurate as FFT
- NU – FFT is more stable than FFT

CONSOB

- NU – FFT allows the use of Gaussian Grids
- NU – FFT is indifferent to Nyquist _Shannon Limit
- NU – FFT is at least as accurate as FFT
- NU – FFT is more stable than FFT
- NU – FFT speed performances are indistinguishable from FFT's ones

NU – FFT
is a natural candidate for
operational use on trading desks

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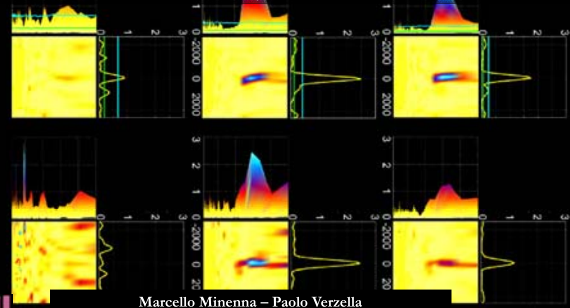

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