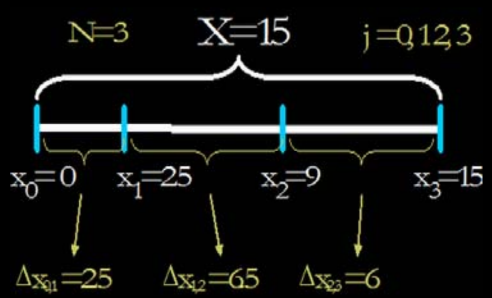
Theorems of Equivalence

The Call Price computed via Convergence Theorem is equal to the Call Price computed via Trapezoid/Simpson Quadrature Rule

- Option Pricing via DFT
  - FT Pricing formula
  - DFT Convergence to FT
  - Convergence Theorems for Uniform Grids
  - Convergence Theorems for Non Uniform Gaussian Grids

Condition 1

Non Uniform Discretization Grid



Condition 1

Gaussian Grids

Optimal choice of discretization points

Gauss Laguerre

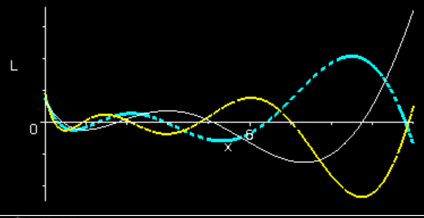
Gander Gautschi

Condition 1

Gaussian Grids

Optimal choice of discretization points

Gauss Laguerre



Zeros of
Laguerre Poynomials

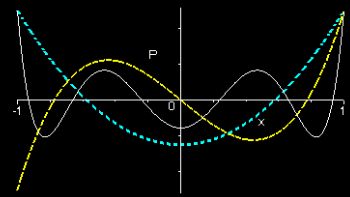
Condition 1

Gaussian Grids

Optimal choice of discretization points

Gauss Lobatto

Zeros of
Legendre Poynomials



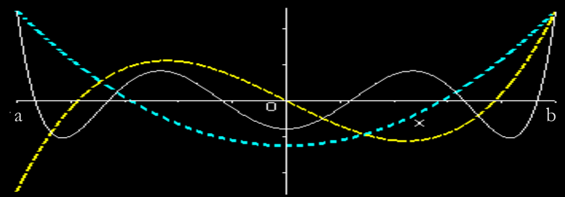
Condition 1

Gaussian Grids

Optimal choice of discretization points

Gander Gautschi

Zeros of rescaled
Legendre Poynomials



Condition 2

N≠M

General DFT

$$\omega(m) = \sum_{j=0}^{N-1} e^{-i\frac{2\pi}{X}x_j(m-1)} f(x_j) \qquad \omega\eta\varepsilon\rho\varepsilon\ \mu=1,2,\square,M$$

The Convergence Theorem (C Th)

$$\mathcal{F}[f(x)](t_m) = \lim_{N \to \infty} \frac{X}{N} \omega(m)$$

$$t_m = \frac{2\pi}{X}(m-1)$$

$$C_0 = \frac{e^{-\alpha \ln K}}{\pi} \int_0^{+\infty} \Re\left(e^{i\xi \ln K} \frac{e^{-rT} f_2(\xi - (\alpha+1)i}{\alpha^2 + \alpha - \xi^2 + i(2\alpha+1)\xi} \right) d\xi$$

Gaussian Grids for *f*

1. $f(\upsilon_{j-1}) = e^{\left[1+i\left(\frac{M\pi}{a*} - \ln S_t\right)\right]\upsilon_{j-1}} \psi_0(\upsilon_{j-1}) \quad \frac{1}{L_{N+1}(\upsilon_{j-1}) L'_N(\upsilon_{j-1})}$
2. $f(\tfrac{1}{2}a(1+\upsilon_{j-1})) = e^{\left[1+i\left(\frac{M\pi}{a*} - \ln S_t\right)\right]\frac{1}{2}a(1+\upsilon_{j-1})} \psi_0(\tfrac{1}{2}a(1+\upsilon_{j-1})) \quad \frac{1}{[P_{N-1}(\upsilon_{j-1})]^2}$

1.

$$C_0 = \frac{e^{-\alpha \ln K}}{\pi} \int_0^{+\infty} \Re\left(e^{i\xi \ln K} \frac{e^{-rT} f_2(\xi - (\alpha+1)i}{\alpha^2 + \alpha - \xi^2 + i(2\alpha+1)\xi} \right) d\xi$$

$$f(\upsilon_{j-1}) = e^{\left[1+i\left(\frac{M\pi}{a*} - \ln S_t\right)\right]\upsilon_{j-1}} \psi_0(\upsilon_{j-1}) \quad \frac{1}{L_{N+1}(\upsilon_{j-1}) L'_N(\upsilon_{j-1})}$$

\+

C-Th

$$C_0([\ln K]_u^*) \approx -\Re\left[\frac{e^{-\alpha\left(\ln S_t - \frac{M\pi}{a*} + \frac{2\pi}{a*}(u-1)\right)}}{\pi} \frac{1}{N+1} \cdot \omega^*(u) \right]$$

2.

$$C_0 = \frac{e^{-\alpha \ln K}}{\pi} \int_0^{+\infty} \Re\left(e^{i\xi \ln K} \frac{e^{-rT} f_2(\xi - (\alpha+1)i}{\alpha^2 + \alpha - \xi^2 + i(2\alpha+1)\xi} \right) d\xi$$

$$f(\tfrac{1}{2}a(1+\upsilon_{j-1})) = e^{\left[1+i\left(\frac{M\pi}{a*} - \ln S_t\right)\right]\frac{1}{2}a(1+\upsilon_{j-1})} \psi_0(\tfrac{1}{2}a(1+\upsilon_{j-1})) \quad \frac{1}{[P_{N-1}(\upsilon_{j-1})]^2}$$

\+

C-Th

$$C_0([\ln K]_u^*) \approx \Re\left[\frac{e^{-\alpha\left(\ln S_t - \frac{M\pi}{a*} + \frac{2\pi}{a*}(u-1)\right)}}{\pi} \frac{1}{N(N-1)} \cdot \omega^*(\tfrac{1}{2}a(1+\upsilon_{j-1})) \right]$$

Theorems of Equivalence

The Call Price computed via Convergence Theorem is equal to the Call Price computed via Gauss Laguerre/Gander Gautschi Quadrature Rule

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    - Gaussian Gridding: a matter of interpolation
    - The Computational Framework: Speed, Stability, Accuracy
- Conclusions

$\overrightarrow{C_t}$ via DFT

allows

Fast Fourier Trasform Algorithms

$\overrightarrow{C_t}$ via DFT

Newton-Cotes

Uniform FFT

$\overrightarrow{C_t}$ via DFT

Gauss

NonUniform FFT

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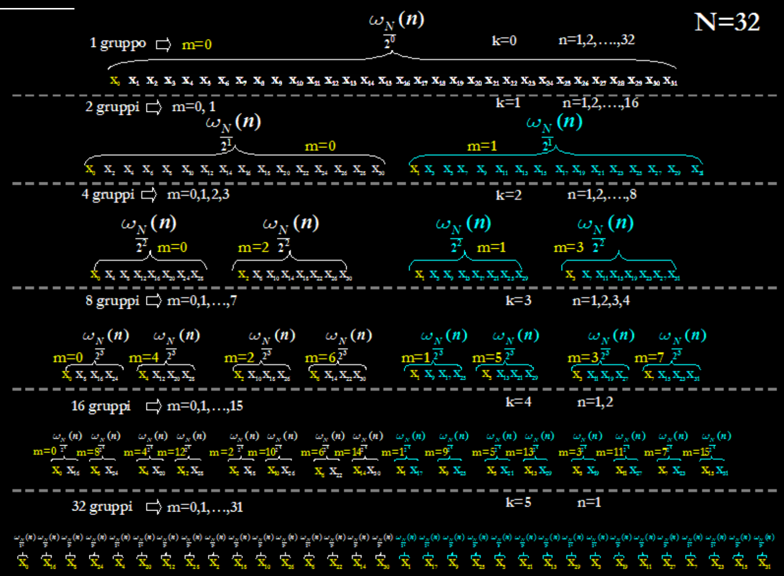
Cooley-Tukey DFT Characterization

$$\omega_2^m(n) = f(x_m) + W_2^{(n-1)} f(x_{m+N/2}) \quad for\ n = 1,2$$

Iterated Bottom – Up for N stages

It gives the FFT Cooley – Tukey Algorithm



FFT Cooley – Tukey Algorithm

The DFT computational cost drops

from

$O\left(N^2\right)$

to

$O\left(N \log_2 N\right)$

Since the Nyquist – Shannon Limit, the pricing formulas

via FFT

Give accurate prices

ONLY

Around the Nyquist Frequency

Since the Nyquist – Shannon Limit, the pricing formulas

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Around the Nyquist Frequency

Approx. 25% of prices can be accepted

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Gaussian Gridding

Gaussian Gridding

Step 1

Gaussian Gridding

Step 1

Gaussian Projection of the non uniformly sampled characteristic function on a oversampled uniform grid

Gaussian Gridding

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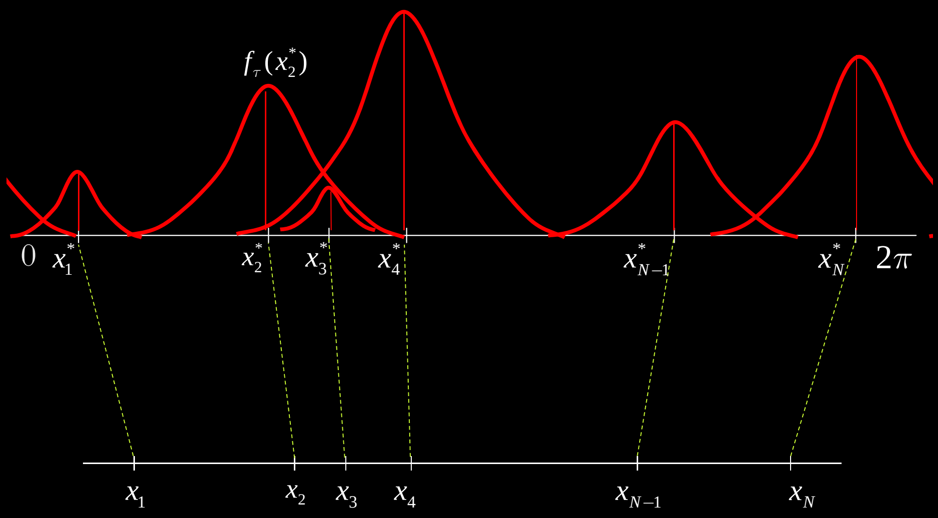
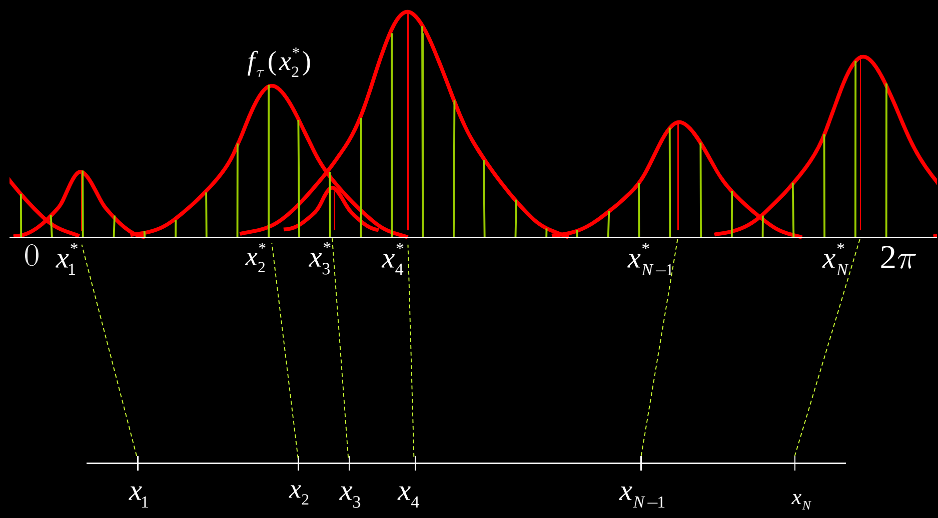
Gaussian Projection of the non uniformly sampled characteristic function on a oversampled uniform grid

$$f_\tau(x) = \sum_{j=0}^{N-1} f(x_j) \sum_{k=-\infty}^{\infty} e^{-\frac{(x_j - x - 2k\pi)^2}{4\tau}}$$





Gaussian Gridding

Step 2

Gaussian Gridding

Step 2

FFT computation on the oversampled grid of the Fourier Coefficient of the reprojected characteristic function

Gaussian Gridding

Step 2

FFT computation on the oversampled grid of the Fourier Coefficient of the reprojected characteristic function

$$F_\tau(n) = \frac{1}{2\pi}\int_0^{2\pi} f_\tau(x)e^{-ix(n-1)}dx$$

Gaussian Gridding

Step 3

Elimination of frequencies greater than Nyquist – Shannon Limit

Gaussian Gridding

Step 4

homothetic rescaling from Gaussian scale

Gaussian Gridding

Step 4

homothetic rescaling from Gaussian scale

$$\omega(n) = \sqrt{\frac{\pi}{\tau}}\, e^{n^2\tau} F_\tau(n)$$

Computational Cost

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The major computational cost of the Procedure is the FFT on the oversampled grid

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Choosing the oversampling ratio

$$M_\tau = 2M$$

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Choosing the oversampling ratio

$$M_\tau = 2M$$

The total cost of the procedure is $\simeq 2M \log_2 2M$

Syllabus of the presentation

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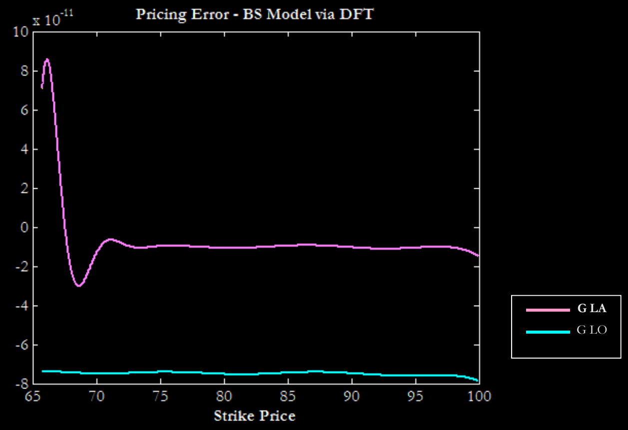
CONSOB

The Computational Framework

ACCURACY

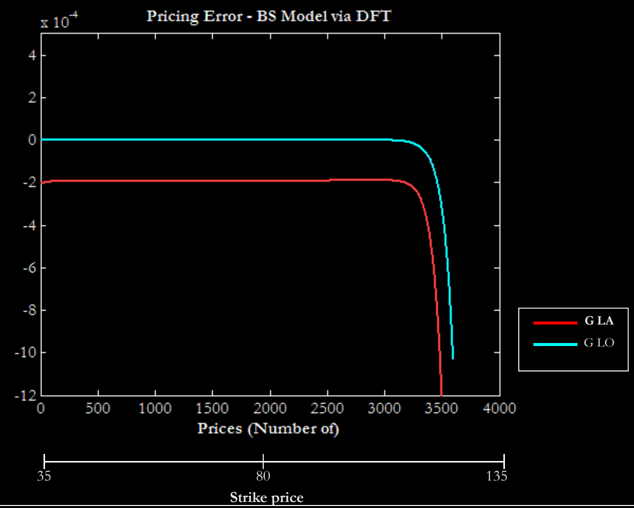


2000 Prices computed

CONSOB

The Computational Framework

STABILITY



CONSOB

The Computational Framework

STABILITY

The error of 90% of prices computed lies in the

CONSOB

STABILITY

The error of 90% of prices computed lies in the

10^{-3}

RANGE OF PRECISION

SPEED

the NU – FFT is around 2 time slower than FFT

SPEED

At very low time scales, the differences disappear

SPEED

At very low time scales, the differences disappear

	NC2	G – LA	G - LO
FFT	0.01 sec.	N/A	N/A
	NC2	G - LA	G - LO
NU – FFT	0.02 sec.	0.0261 sec.	0.0301 sec.

Computation of 4000 prices on a Centrino 1600Mhz – 2gb RAM
Mean Value over 1000 runs

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Conclusions

- NU – FFT allows the use of Gaussian Grids

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- NU – FFT is at least as accurate as FFT
- NU – FFT is more stable than FFT

Conclusions

- NU – FFT allows the use of Gaussian Grids
- NU – FFT is indifferent to Nyquist _Shannon Limit
- NU – FFT is at least as accurate as FFT
- NU – FFT is more stable than FFT
- NU – FFT speed performances are indistinguishable from FFT's ones

Conclusions

NU – FFT

is a natural candidate for operational use on trading desks