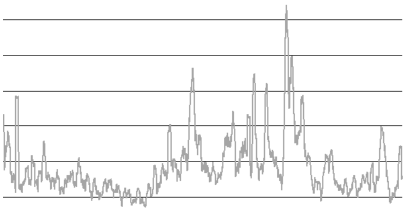
Estimating Volatility Through GARCH Diffusive Approach



Marcello Minenna – Giovanna Maria Boi
International Summer School on Risk Measurement and Control,
University of Tor Vergata, Rome 16th June 2007

Syllabus of the presentation

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- **Volatility: a Crucial Tool in Mathematical Finance**
- **Forecasting Volatility through GARCH Diffusive Approach**
  - The Convergence Theorem on R^2
  - The Diffusion Limit of M-GARCH(1,1)
  - The Predictive Interval for the Volatility
  - Other GARCH Processes
- **GARCH Diffusive Approach vs Mutual Funds' Risk Assessment**
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- **Conclusions**

Volatility: a crucial Tool in Mathematical Finance

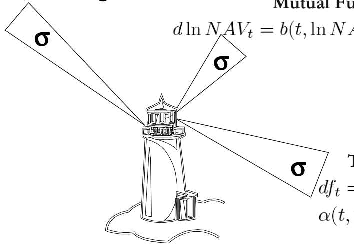
VOLATILITY IS USEFULLY EMPLOYED IN SEVERAL MATHEMATICAL FINANCE PROBLEMS, LIKE…

Derivatives pricing

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

Mutual Funds Risk Assessment

$$d \ln NAV_t = b(t, \ln NAV_t)dt + \sigma(t, \ln NAV_t) dW_t$$



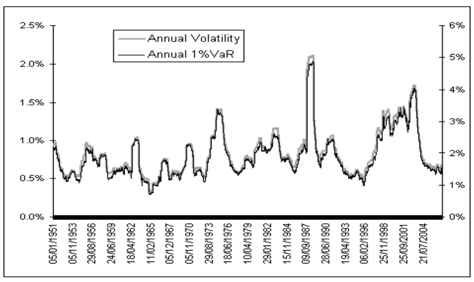
Term Structure Modelling

$$df_t = \alpha(t,T)dt + \sigma(t,T)dW_t$$

$$\alpha(t,T)dt = \sigma(t,T)\int_t^T \sigma(t,s)ds$$

Volatility: a crucial Tool in Mathematical Finance

VOLATILITY HAS A CLOSE CORRESPONDENCE WITH ANY RISK MEASURE, LIKE VaR AND EXPECTED SHORTFALL



Volatility: A Crucial Tool in Mathematical Finance

VOLATILITY ESTIMATES CAN BE OBTAINED IN TWO ALTERNATIVE WAYS:

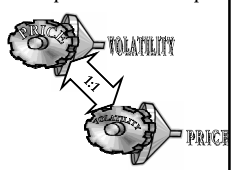
Method 1: Subjective Prob. Meas.

σ = the standard deviation of historical observations

$$\sigma = \sqrt{\sum_{i=1}^{T} \frac{(X_i - \overline{X})^2}{T-1}}$$

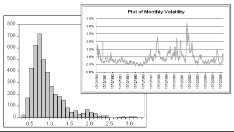
Method 2: Risk-Neutral Prob. Meas.

σ = the value which equals the Risk-Neutral price to the market price



Volatility: a crucial Tool in Mathematical Finance

VOLATILITY VARIES
RANDOMLY OVER TIME



NEED FOR VOLATILITY FORECASTS BASED ON STOCHASTIC VOLATILITY MODELS

TIME SERIES ANALYSIS OF VOLATILITY

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Forecasting Volatility through GARCH Diffusive Approach

MODELLING THE TIME SERIES OF HISTORICAL VOLATILITIES THROUGH THE DIFFUSION LIMIT OF GARCH PROCESSES

from: STOCHASTIC DIFFERENCE EQUATIONS

to: STOCHASTIC DIFFERENTIAL EQUATIONS

via: SHRINKING *of* TIME INTERVALS

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Forecasting Volatility through GARCH Diffusive Approach

THE CONVERGENCE THEOREM on R^2

The sequence $\{X_t^h\}$, whose measurable space is $(\mathbb{R}^2, \mathbb{B}(\mathbb{R}^2))$, converges weakly for h $\downarrow$0 to the process $\{X_t\}$ which has a unique distribution and is characterized by the following stochastic differential equation:

$$dX_t = b(x,t)dt + \sigma(x,t)dW_{2,t}$$

where $W_{2,t}$ is a two-dimensional standard Brownian motion, if the conditions 1-4, presented below, are satisfied.

Forecasting Volatility through GARCH Diffusive Approach

THE CONVERGENCE THEOREM on R^2
(cont'd)

More explicitly, the sequence $\{X_t^h\}$, composed by the sequences $\{X_{1,t}^h\}$ and $\{X_{2,t}^h\}$, each of them measurable on the space $(\mathbb{R}^1, \mathbb{B}(\mathbb{R}^1))$, converges weakly for h $\downarrow$0 to the following system of stochastic differential equations:

$$\begin{aligned} dX_{1,t} &= b(x_1,t)dt + \sigma(x_1,t)dW_t \\ dX_{2,t} &= b(x_2,t)dt + \sigma(x_2,t)dW_t^* \end{aligned}$$

where W_t and W_t^* are two independent uni-dimensional standard Brownian motions, and $X_{1,t}$ and $X_{2,t}$ are two independent processes which take values on $\mathbb{R}^1$, if the conditions 1-4, presented below, are satisfied.

Forecasting Volatility through GARCH Diffusive Approach

THE CONVERGENCE THEOREM on R²
(cont'd)

The process $\{X_t\}$ has a distribution independent on the choice of $\sigma(x,t)$ and it takes finite values over finite time intervals, i.e. $\forall T > 0$:

$$P(\sup_{0\leq t\leq T} \|X_t\| < \infty) = 1$$

Forecasting Volatility through GARCH Diffusive Approach

THE CONVERGENCE THEOREM on R²
(cont'd)

CONDITION 1

If there exists a $\delta>0$ such that:

$$\lim_{h\downarrow 0} \begin{pmatrix} c_{h,\delta}(x_1,t) \\ c_{h,\delta}(x_2,t) \end{pmatrix} = 0$$

Then there exist $a(x,t)$ and $b(x,t)$, continuous measures respectively mapping from $\mathbb{R}^2 \times [0,\infty)$ into the space of the *2x2* semi-definite positive matrices, and from $\mathbb{R}^2 \times [0,\infty$ into $\mathbb{R}^2$, such that:

$$\lim_{h\downarrow 0}\begin{pmatrix} b_h(x_1,t) \\ b_h(x_2,t) \end{pmatrix} = \begin{pmatrix} b(x_1,t) \\ b(x_2,t) \end{pmatrix}$$

$$\lim_{h\downarrow 0} \begin{pmatrix} a_h(x_1,t) & a_h((x_1,x_2),t) \\ a_h((x_2,x_1),t) & a_h(x_2,t) \end{pmatrix} = \begin{pmatrix} a(x_1,t) & 0 \\ 0 & a(x_2,t) \end{pmatrix}$$

Forecasting Volatility through GARCH Diffusive Approach

THE CONVERGENCE THEOREM on R²
(cont'd)

CONDITION 2

There exists $\sigma(x,t)$, a continuous mapping from $\mathbb{R}^2 \times [0,\infty)$ into $\mathbb{R}^2$, such that, $\forall x_1 \in \mathbb{R}^1, \forall x_2 \in \mathbb{R}^1$, it holds:

$$\begin{pmatrix} \sigma(x_1,t) & 0 \\ 0 & \sigma(x_2,t) \end{pmatrix} = \begin{pmatrix} \sqrt{a(x_1,t)} & 0 \\ 0 & \sqrt{a(x_2,t)} \end{pmatrix}$$

Forecasting Volatility through GARCH Diffusive Approach

THE CONVERGENCE THEOREM on R²
(cont'd)

CONDITION 3

For $h\downarrow 0$, X_0^h converges in distribution to a random variable X_0 with probability measure v_0 on $(\mathbb{R}^2, \mathbb{B}(\mathbb{R}^2))$

CONDITION 4

v_0 , $a(x,t)$ and $b(x,t)$ uniquely specify the distribution of the process $\{X_t\}$ characterized by an initial distribution v_0 , a conditional second moment $a(x,t)$ and a conditional first moment $b(x,t)$

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Forecasting Volatility through GARCH Diffusive Approach

THE DIFFUSION LIMIT OF THE M-GARCH(1,1):
Statement

Given the M-GARCH(1,1) model:

$$\begin{cases} X_k - X_{k-1} = \gamma \cdot (\eta - X_{k-1}) + \sigma_k \widetilde{Z}_k \\ \text{and} \\ \ln \sigma^2_{k+1} - \ln \sigma^2_k = \beta_0^{(k)} + (\beta_1^{(k)} - 1)\ln \sigma_k^2 + \beta_1^{(k)} \ln Z_k^2 \\ \text{or, equivalently} \\ \ln \sigma^2_{k+1} - \ln \sigma^2_k = \beta_0^{(k)} + (\beta_1^{(k)} - 1)\ln \sigma_k^2 + 2\beta_1^{(k)} \ln |Z_k| \end{cases}$$

$\widetilde{Z}_k$ and Z_k are i.i.d. *N(0,1)*

its diffusion limit is:

$$dX_t = q(\mu - X_t)dt + \sigma_t dW_t$$

$$d\ln\sigma_t^2 = \left(\beta_0 + 2\beta_1 E(\ln|Z_t|) + (\beta_1 - 1)\ln\sigma_t^2\right)dt + 2\beta_1\sqrt{Var(\ln|Z_t|)}dW_t^*$$

Z_t is *N(0,1)*

Forecasting Volatility through GARCH Diffusive Approach

THE DIFFUSION LIMIT OF THE M-GARCH(1,1):
Proof[1]

STEP 1:
THE RE-SCALING OF THE PROCESS

The k intervals are divided into $1/h$ subintervals each one of length h

⇩

$$\ln \sigma^2_{(k+1)h} - \ln \sigma^2_{kh}$$
$$=$$
$$\beta_{0h} + (\beta_{1h} - h)\ln \sigma^2_{kh} + 2\beta_{1h}\left\{\sqrt{h}\left[\ln|Z_k| - E\left(\ln|Z_k|\right)\right] + E\left(\ln|Z_k|\right)\right\}$$

[1] The focus is on the difference equation for the volatility. For the convergence of the first equation see Minenna 2001.

Forecasting Volatility through GARCH Diffusive Approach

THE DIFFUSION LIMIT OF THE M-GARCH(1,1):
Proof (cont'd)

STEP 2:
THE CONSTRUCTION OF THE PROCESS $\left\{\ln \sigma_t^{2^h}\right\}$

Definition of the probability measure P_h on the Skorokhod Space D such that:

$$P_h(\ln \sigma_0^{2^h} \in \Gamma) = v_0(\Gamma) \qquad \forall \Gamma \in \mathbb{B}(\mathbb{R}^1)$$
$$P_h(\ln \sigma_t^{2^h} = \ln \sigma^2_{kh}, \quad \forall\, kh \le t < (k+1)h) = 1$$
$$P_h(\ln \sigma^2_{(k+1)h} \in \Gamma | \widehat{\Im}_{kh}) = \Pi_{h,kh}(\ln \sigma^2_{kh}, \Gamma) \text{ a.s. under } P_h,\ \forall k \ge 0,\ \forall \Gamma \in \mathbb{B}(\mathbb{R}^1)$$

⇩

$$\ln \sigma_{t+1}^{2^h} - \ln \sigma_t^{2^h}$$
$$=$$
$$\beta_{0h} + (\beta_{1h} - h)\ln \sigma_t^{2^h} + 2\beta_{1h}\left\{\sqrt{h}\left[\ln|Z_t^h| - E\left(\ln|Z_t^h|\right)\right] + E\left(\ln|Z_t^h|\right)\right\}$$

Forecasting Volatility through GARCH Diffusive Approach

THE DIFFUSION LIMIT OF THE M-GARCH(1,1):
Proof (cont'd)

STEP 3:
CHECK OF CONDITION 1 OF THE CONVERGENCE THEOREM

Finding the values of β_{0h} **and** β_{1h} **which guarantee the convergence of the conditional moments**

$$\left.\begin{array}{l}\beta_{0h} := \beta_0 \cdot h \\ \beta_{1h} := \beta_1 \cdot h\end{array}\right\} \Longleftrightarrow \begin{cases}\lim\limits_{h\downarrow 0} c_{h,1}\left(\ln \widehat{\sigma^2}, t\right) = 0 \\ \lim\limits_{h\downarrow 0} b_h\left(\ln \widehat{\sigma^2}, t\right) = \beta_0 + 2\beta_1 E\left(\ln|Z_t|\right) + (\beta_1 - 1)\ln \sigma_t^2 \\ \lim\limits_{h\downarrow 0} a_h\left(\ln \widehat{\sigma^2}, t\right) = 4\beta_1^2 Var\left(\ln|Z_t|\right)\end{cases}$$

Forecasting Volatility through GARCH Diffusive Approach

THE DIFFUSION LIMIT OF THE M-GARCH(1,1):
Proof (cont'd)

STEP 4:
CHECK OF CONDITIONS 2, 3 and 4 OF THE CONVERGENCE THEOREM

⇩

- **Condition 2 is verified for every σ >0, i.e.:**

$$\sigma\left(\ln \widehat{\sigma^2}, t\right) = 2\beta_1 \sqrt{Var(|Z_t|)}$$

- **Condition 3 is evidently satisfied by construction of the process** $\left\{\ln \sigma_t^{2^h}\right\}$
- **Consequently, Condition 4 is verified too.**

Q.E.D.

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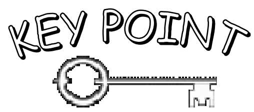
Forecasting Volatility through GARCH Diffusive Approach



From the Diffusion Limit of the GARCH Process it is possible to establish a *Predictive Interval for* σ^2_t

THE PREDICTIVE INTERVAL
The properties of the Stochastic Differential Equation
in the case of the M-GARCH(1,1)

$$d\ln\sigma_t^2 = \left(\beta_0 + 2\beta_1 E(\ln|Z_t|) + (\beta_1 - 1)\ln\sigma_t^2\right)dt + 2\beta_1\sqrt{Var(\ln|Z_t|)}dW_t^*$$

$$\ln\sigma_t^2 \sim N\left[\left(\ln\sigma_{t-1}^2 + \frac{\beta_0+2\beta_1 E(\ln|Z_t|)}{(\beta_1-1)}\right)e^{(\beta_1-1)} - \frac{\beta_0+2\beta_1 E(\ln|Z_t|)}{(\beta_1-1)};\sqrt{\frac{\left(2|\beta_1|\sqrt{Var(\ln|Z_t|)}\right)^2}{2(\beta_1-1)}\left(e^{2(\beta_1-1)}-1\right)}\right]$$

THE PREDICTIVE INTERVAL
The estimation of the Parameters of the Stochastic Differential Equation

The relationship between the Stochastic Difference Equation and the Stochastic Differential Equation

$$\ln\sigma_{k+1}^2 - \ln\sigma_k^2 = \beta_0^{(k)} + (\beta_1^{(k)} - 1)\ln\sigma_k^2 + 2\beta_1^{(k)}\ln|Z_k|$$

$$d\ln\sigma_t^2 = \left(\beta_0 + 2\beta_1 E(\ln|Z_t|) + (\beta_1 - 1)\ln\sigma_t^2\right)dt + 2\beta_1\sqrt{Var(\ln|Z_t|)}dW_t^*$$

THE PREDICTIVE INTERVAL
The estimation of the Parameters of the Stochastic Differential Equation
(cont'd)

Matching of the first two Conditional Moments

$$\ln\sigma_{k+1}^2 - \ln\sigma_k^2$$
$$=$$
$$\left(e^{(\beta_1-1)}-1\right)\left(\frac{\beta_0+2\beta_1 E(\ln|Z_t|)}{(\beta_1-1)}\right) + \left(e^{(\beta_1-1)}-1\right)\ln\sigma_k^2 + 2\left(e^{(\beta_1-1)}-1\right)\ln|Z_k|$$

THE PREDICTIVE INTERVAL
The estimation of the Parameters of the Stochastic Differential Equation
(cont'd)

The Maximum Likelihood Method

$$\ln\sigma_{k+1}^2 - \ln\sigma_k^2 = \widehat{a} + \widehat{b}\ln\sigma_k^2 + e_k$$

$$\beta_0 = f_1\left(\widehat{a},\widehat{b}\right)$$

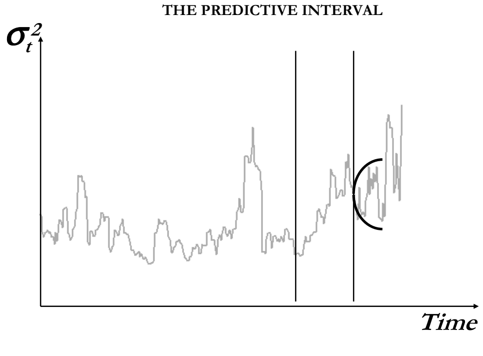
$$\beta_1 = f_2\left(\widehat{a},\widehat{b}\right)$$

$$2\beta_1\sqrt{Var(\ln|Z_t|)} = f_3\left(\widehat{a},\widehat{b},e_k\right)$$

THE PREDICTIVE INTERVAL:
The Normality Prediction Interval for the Volatility

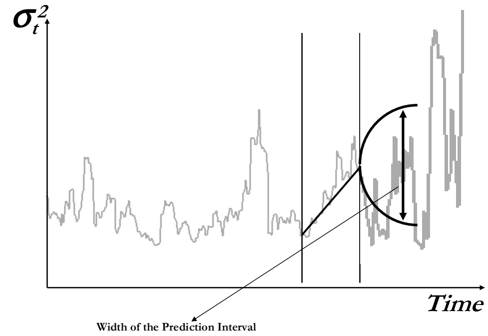
$$P\left(\begin{array}{c}-z_{\frac{\alpha}{2}}\sqrt{\frac{\left(2|\beta_1|\sqrt{Var(\ln|Z_t|)}\right)^2}{2(\beta_1-1)}\left(e^{2(\beta_1-1)}-1\right)} + \left(\ln\sigma_{t-1}^2 + \frac{\beta_0+2\beta_1 E(\ln|Z_t|)}{(\beta_1-1)}\right)e^{(\beta_1-1)} - \frac{\beta_0+2\beta_1 E(\ln|Z_t|)}{(\beta_1-1)} \\ \le \ln\sigma_t^2 \le \\ -z_{\frac{\alpha}{2}}\sqrt{\frac{\left(2|\beta_1|\sqrt{Var(\ln|Z_t|)}\right)^2}{2(\beta_1-1)}\left(e^{2(\beta_1-1)}-1\right)} + \left(\ln\sigma_{t-1}^2 + \frac{\beta_0+2\beta_1 E(\ln|Z_t|)}{(\beta_1-1)}\right)e^{(\beta_1-1)} - \frac{\beta_0+2\beta_1 E(\ln|Z_t|)}{(\beta_1-1)}\end{array}\right) = \alpha$$

THE PREDICTIVE INTERVAL



Forecasting Volatility through GARCH Diffusive Approach

THE PREDICTIVE INTERVAL



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Forecasting Volatility through GARCH Diffusive Approach

OTHER GARCH PROCESSES
Analogous Procedure

THE DIFFUSION LIMIT OF THE E-GARCH(1,1)

Given the E-GARCH(1,1) model:

$$\ln \sigma_{k+1}^2 - \ln \sigma_k^2 \quad = \quad \beta_0^{(k)} + \beta_1^{(k)} \ln \sigma_k^2 + \beta_2^{(k)} \left(|Z_k| + \vartheta Z_k\right)$$

Z_k is *N(0,1)*

its diffusion limit is:

$$d \ln \sigma_t^2 \quad = \quad \left[\alpha_0 + \frac{2}{\sqrt{2\pi}} \left(\alpha_4 + \frac{\alpha_5}{2}\right) - \alpha_4 - \alpha_5 - \alpha_1 - 1 + (\alpha_1 - 1) \ln \sigma_t^2\right] dt - \frac{\alpha_5}{2} dW_t + \left|\alpha_4 + \frac{\alpha_5}{2}\right| \sqrt{\frac{\pi - 2}{\pi}} dW_t^*$$

THE DIFFUSION LIMIT OF THE L-GARCH(1,1

Given the E-GARCH(1,1) model:

$$\sigma_{k+1}^2 - \sigma_k^2 = \omega + \sigma_k^2(\beta + \varpi Z_k^2 - 1)$$

Z_k is *N(0,1)*

its diffusion limit is:

$$d\sigma_t^2 = [\omega + \vartheta \sigma_t^2] dt + \sqrt{2\varpi} \sigma_t^2 dW_t$$

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GARCH Diffusive Approach vs Mutual Funds' Risk Assessment

MUTUAL FUNDS: PRELIMINARY CONCEPTS

$$Fund_t = \sum_{j=1}^{J} x_{j,t} A_{j,t}$$

⇩

$$NAV_t = \frac{\sum_{j=1}^{J} x_{j,t} A_{j,t}}{S_t^P}$$

S_t^P= number of fund's oustanding shares at time *t*
hp.:S_t^P = 1

MUTUAL FUNDS: PRELIMINARY CONCEPTS
(cont'd)

Net Asset Value
can be autonomously modeled
as a Random Variable

Net Asset Value Time Series
becomes
a Stochastic Process

MUTUAL FUNDS: PRELIMINARY CONCEPTS
(cont'd)

$$r_t^{NAV} = \frac{NAV_t - NAV_s}{NAV_s}$$

or, equivalently, for $t \downarrow s$

$$r_t^{NAV} = \ln\left(\frac{NAV_t}{NAV_s}\right)$$

The Time Series of NAV returns
becomes
a Stochastic Process

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THE ITALIAN REGULATION FOR THE PROSPECTUS OF MUTUAL FUNDS

Any mutual fund has to declare on yearly basis its risk undertaking, i.e. the qualitative risk-class which it belongs to.

Fund Risk Classes
low
medium-low
medium
medium-high
high
very high

"The risk level has to be indicated in the Prospectus in descriptive terms…and it has to be estimated taking as a reference the volatility of the fund's shares … ."

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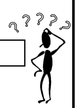
THE DAILY PROBLEM OF THE ASSET MANAGER

What is the position of my portfolio inside the declared risk class?



MAPPING OF EACH QUALITATIVE RISK CLASS INTO ITS "QUANTITATIVE EQUIVALENT" IN ORDER TO ASSESS

THE MIGRATION RISK

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GARCH Diffusive Approach vs Mutual Funds' Risk Assessment

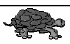
THE "TRADITIONAL" SOLUTION
(from Classical Portfolio Theory)

The marginal contribution of a new asset to the fund's risk

$$\left.\frac{\partial \sigma^2_{new}}{\partial x_{A^{new}}}\right|_{x_{A^{new}}=0} = -2\sigma^2_{old} + 2cov(r_{old}, r_{A^{new}})$$

GARCH Diffusive Approach vs Mutual Funds' Risk Assessment

THE "TRADITIONAL" SOLUTION:
<u>PITFALLS</u>

- **computationally heavy** 
- **conceptually wrong!!!** $\sum_{j=1}^{J} x_{j,t} r_{A_{j,t}} \neq \frac{NAV_t - NAV_s}{NAV_s}$

$\left.\frac{\partial \sigma^2_{new}}{\partial x_{A^{new}}}\right|_{x_{A^{new}}=0}$  $!_d + 2cov(r_{old}, r_{A^{new}})$

Syllabus of the presentation

- **Volatility: a Crucial Tool in Mathematical Finance**
- **Forecasting Volatility through GARCH Diffusive Approach**
  - The Convergence Theorem on R^2
  - The Diffusion Limit of M-GARCH(1,1)
  - The Predictive Interval for the Volatility
  - Other GARCH Processes
- **GARCH Diffusive Approach vs Mutual Funds' Risk Assessment**
  - Mutual Funds: Preliminary Concepts
  - The Italian Regulation for the Prospectus of Mutual Funds
  - The Daily Problem of the Asset Manager
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- **Conclusions**

GARCH Diffusive Approach vs Mutual Funds' Risk Assessment

THE SOLUTION THROUGH THE GARCH DIFFUSIVE APPROACH:
mapping of the qualitative risk classes to calibrated volatility intervals

<u>Step 1</u>: "Arbitrary" Definition of Fund Loss Intervals (e.g. using the VaR)

Fund Risk Classes	Fund Loss Intervals *min* *max*
low	
medium- low	
medium	
medium-high	
high	
very high	

GARCH Diffusive Approach vs Mutual Funds' Risk Assessment

THE SOLUTION THROUGH THE GARCH DIFFUSIVE APPROACH:
mapping of the qualitative risk classes to calibrated volatility intervals

<u>Step 2</u>: Mapping Fund Loss Intervals to corresponding Fund Volatility Intervals derived from the calibration of a suitable GARCH Diffusion Model

Fund Risk Classes	Fund Loss Intervals *min* *max*
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GARCH Diffusive Approach vs Mutual Funds' Risk Assessment

THE SOLUTION THROUGH THE GARCH DIFFUSIVE APPROACH:
mapping of the qualitative risk classes to calibrated volatility intervals

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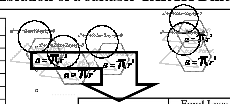
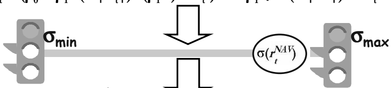


Fund Risk Classes	Fund Loss Intervals *min*	*max*	σ_{min}	σ_{max}
low				
medium-low				
medium				
medium-high				
high				
very high				

GARCH Diffusive Approach vs Mutual Funds' Risk Assessment

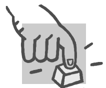
THE SOLUTION THROUGH THE GARCH DIFFUSIVE APPROACH:
mapping of the qualitative risk classes to calibrated volatility intervals

$$d \ln \sigma_t^2 = (\beta_0 + 2\beta_1 E(\ln|Z_t|) + (\beta_1 - 1)\ln \sigma_t^2)\,dt + 2\beta_1 \sqrt{Var(\ln|Zt|)}\; dW_t^*$$



GARCH Diffusive Approach vs Mutual Funds' Risk Assessment

THE SOLUTION THROUGH THE GARCH DIFFUSIVE APPROACH:
mapping of the qualitative risk classes to calibrated volatility intervals



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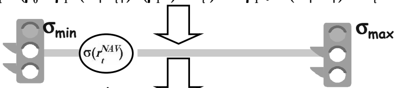

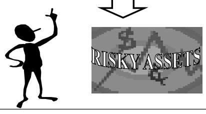

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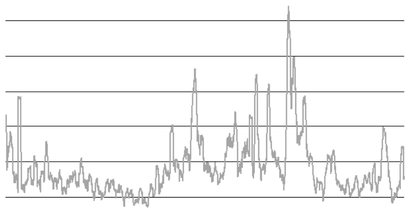
CONCLUSIONS

- **Volatility is a crucial risk measure in finance:**
  - wide range of of applications
  - driver of other risk measures
- **Volatility varies randomly over time**
- **The GARCH Diffusive Approach allows to model the stochastic process of the volatility and to make forecasts based on reliable prediction intervals**
- **The GARCH Diffusive Approach can be usefully implemented in the mutual funds industry to forecast volatility, to evaluate the risk exposure, and, consequently, to select the optimal portfolio allocation given the full knowledge of the risk undertaking associated to each possible alternative**

Estimating Volatility Through GARCH Diffusive Approach



Marcello Minenna – Giovanna Maria Boi
International Summer School on Risk Measurement and Control, University of Tor Vergata, Rome 16th June 2007