10.8.2. Lebesgue integral for non-negative functions

Definition 10.47. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f : X \to [0, +\infty]$ be a non-negative function which belongs to the class $\text{Meas}(X, \mathcal{M}), ([0, +\infty], \overline{\mathbb{B}}_{[0,+\infty]})$. The *abstract integral or Lebesgue integral* of f on X with respect to μ, denoted by $\int_X f(x)\, d\mu(x)$, is defined as

$$\int_X f(x)\, d\mu(x) = \sup_{0 \le s(x) \le f(x)} \int_X s(x)\, d\mu(x) \tag{10.96}$$

and, by convention, $0 \times \infty = 0$.

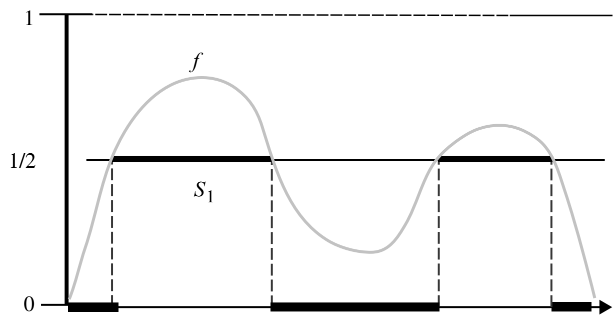
Notation 10.10. Often, instead of denoting the abstract integral of f on X with respect to μ by $\int_X f(x)\, d\mu(x)$, one may write

$$\int_X f(x)\mu\ (dx)$$

or

$$\int_X f\, d\mu$$

Figure 10.2 The construction of the Lebesgue integral – step 1.



Remark 10.7. Under Definition 10.47, the Lebesgue integral of a measurable non-negative function is a non-negative quantity, ie,

$$\int_X f(x)\, d\mu(x) \geq 0 \tag{10.97}$$

More specifically, it may be

$$\int_X f(x)\, d\mu(x) = +\infty \tag{10.98}$$

Remark 10.8 (Intuitive interpretation of the Lebesgue integral). Figures 10.2–10.5 provide an intuitive interpretation about the meaning of Equation (10.96) which defines the Lebesgue integral of a measurable non-negative function f. As a matter of fact, they show that this integral is actually the integral of the boundary of the sequence of simple functions $\{s_n\}_{n\in\mathbb{N}}$, which, under Equation (10.78), is punctually and monotonically convergent from below to function f.

Notation 10.11. Often, given the complete measure space $(\mathbb{R}, \widetilde{\mathbb{B}}, \widetilde{m}_1)$ and given a non-negative function $f : \mathbb{R} \to [0, +\infty]$ which belongs to class $\text{Meas}((\mathbb{R}, \widetilde{\mathbb{B}}), ([0, +\infty], \overline{\mathbb{B}}_{[0,+\infty]}))$, the abstract integral of f on $\mathbb{R}$ with respect to the Lebesgue measure, $\widetilde{m}_1$, is expressed by denoting dx instead of $d\widetilde{m}_1(x)$, ie,

$$\int_{\mathbb{R}} f(x)\, d\widetilde{m}_1(x) \equiv \int_{\mathbb{R}} f(x)\, dx$$

Proposition 10.52. *Let* $(X, \mathcal{M}, \mu)$ *be a measure space and let* $f : X \to [0, +\infty]$ *be a non-negative function which belongs to the class* $\text{Meas}((X, \mathcal{M}), ([0, +\infty], \overline{\mathbb{B}}_{[0,+\infty]}))$,

Figure 10.3 The construction of the Lebesgue integral – step 2.

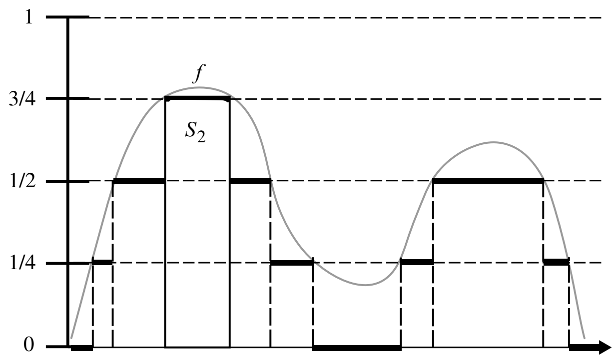


Figure 10.4 The construction of the Lebesgue integral – step 3.

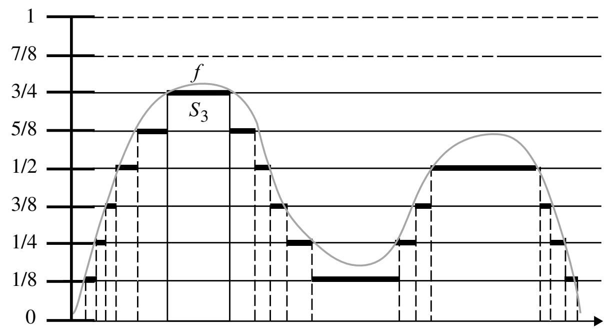
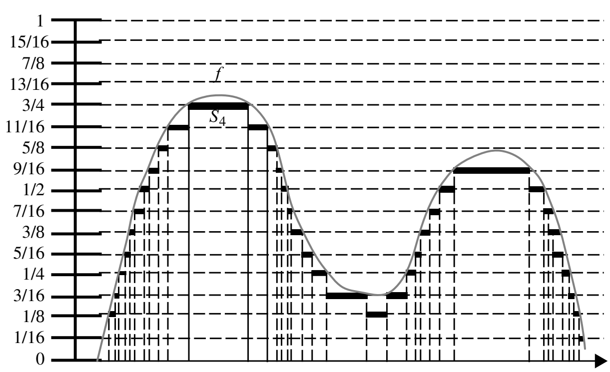


Figure 10.5 The construction of the Lebesgue integral – step 4.



then, given $E \subseteq X$, the following relation holds:

$$\int_X f(x)\, d\mu(x) = \int_E f(x)\, d\mu(x) + \int_{X \setminus E} f(x)\, d\mu(x) \tag{10.99}$$

Definition 10.48. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f : X \to [0, +\infty]$ be a non-negative function which belongs to the class $\text{Meas}((X, \mathcal{M}), ([0, +\infty], \overline{\mathbb{B}}_{[0,+\infty]}))$. For any $E \in \mathcal{M}$, the *abstract integral or Lebesgue integral of f on E with respect to μ*, denoted by $\int_E f(x)\, d\mu(x)$, is defined as

$$\int_E f(x)\, d\mu(x) := \int_X \mathbf{1}_{(E)}(x) f(x)\, d\mu(x) \tag{10.100}$$

and, by convention, $0 \times \infty = 0$.

Remark 10.9. If $f(x) = 1$ for all $x \in E$, $E \in \mathcal{M}$, then one has

$$\int_E 1\, d\mu(x) = \mu(E) \tag{10.101}$$

Proposition 10.53. *Let $(X, \mathcal{M}, \mu)$ be a measure space, let E be a subset of X so that $E \in \mathcal{M}$ and let $f : E \to [0, +\infty]$ be a non-negative function which belongs to the class* $\text{Meas}((E, \mathcal{M}_E), ([0, +\infty], \overline{\mathbb{B}}_{[0,+\infty]}))$. *Also, let $(E, \mathcal{M}_E, \mu_E)$ be the measure space where μ_E is the restriction of measure μ to set E and $\mathcal{M}_E$ is the restriction of the σ-algebra $\mathcal{M}$ to set E* [2]. *Then the abstract integral of f on subset E of X with respect to measure μ coincides with the abstract integral of f on set E with respect to measure μ_E, ie,*

$$\int_E f(x)\, d\mu(x) = \int_E f(x)\, d\mu_E(x) \tag{10.102}$$

Corollary 10.18. *Let $(X, \mathcal{M}, \mu)$ be a measure space, let E be a subset of X so that $E \in \mathcal{M}$ and let $f : E \to [0, +\infty]$ be a non-negative function which belongs to the class* $\text{Meas}((E, \mathcal{M}_E), ([0, +\infty], \overline{\mathbb{B}}_{[0,+\infty]}))$. *Then, if $\mu(E) = 0$, one has*

$$\int_E f(x)\, d\mu(x) = 0 \tag{10.103}$$

[2] Alternatively, μ_E is the measure defined on the σ-algebra $\mathcal{M}_E$ as in Equation (10.5); see Definitions 1.100 and 10.6, Notation 10.1 and Proposition 10.2